

The competition-common enemy graphs of digraphs satisfying Conditions $C(p)$ and $C'(p)$

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Abstract

S. -R. Kim and F. S. Roberts (2002) introduced the following conditions $C(p)$ and $C'(p)$ for digraphs as generalizations of the condition for digraphs to be semiorders. The condition $C(p)$ (resp. $C'(p)$) is: For any set S of p vertices in D , there exists $x \in S$ such that $N_D^+(x) \subseteq N_D^+(y)$ (resp. $N_D^-(x) \subseteq N_D^-(y)$) for all $y \in S$, where $N_D^+(x)$ (resp. $N_D^-(x)$) is the set of out-neighbors (resp. in-neighbors) of x in D . The *competition graph* of a digraph D is the (simple undirected) graph which has the same vertex set as D and has an edge between two distinct vertices x and y if $N_D^+(x) \cap N_D^+(y) \neq \emptyset$. Kim and Roberts characterized the competition graphs of digraphs which satisfy Condition $C(p)$.

The *competition-common enemy graph* of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices x and y if it holds that both $N_D^+(x) \cap N_D^+(y) \neq \emptyset$ and $N_D^-(x) \cap N_D^-(y) \neq \emptyset$. In this note, we characterize the competition-common enemy graphs of digraphs satisfying Conditions $C(p)$ and $C'(p)$.

Keywords: competition-common enemy graph; semiorder; interval order; Condition $C(p)$

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1. Introduction

J. E. Cohen [2] introduced the notion of a competition graph in 1968 in connection with a problem in ecology. The *competition graph* $C(D)$ of a digraph D is the (simple undirected) graph $G = (V, E)$ which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if $N_D^+(x) \cap N_D^+(y) \neq \emptyset$, where $N_D^+(x) := \{v \in V(D) \mid (x, v) \in A(D)\}$ is the set of out-neighbors of x in D . It has been one of important research problems in the study of competition graphs to characterize the competition graphs of digraphs satisfying some conditions.

Definition. A digraph $D = (V, A)$ is called a *semiorder* if there exist a real-valued function $f : V \rightarrow \mathbb{R}$ on the set V and a positive real number $\delta \in \mathbb{R}$ such that $(x, y) \in A$ if and only if $f(x) > f(y) + \delta$.

A digraph $D = (V, A)$ is called an *interval order* if there exists an assignment $J : V \rightarrow 2^{\mathbb{R}}$ of a closed real interval $J(x) \subset \mathbb{R}$ to each vertex $x \in V$ such that $(x, y) \in A$ if and only if $\min J(x) > \max J(y)$. \square

Kim and Roberts characterized the competition graphs of semiorders and interval orders as follows:

Theorem 1.1 ([3]). *Let G be a graph. Then the following are equivalent.*

- (a) *G is the competition graph of a semiorder,*
- (b) *G is the competition graph of an interval order,*
- (c) *$G = K_r \cup I_q$ where if $r \geq 2$ then $q \geq 1$.* \square

Moreover, Kim and Roberts [3] introduced some conditions, which are called Condition $C(p)$ and Condition $C'(p)$, for digraphs as generalizations of the condition for digraphs to be semiorders, and they gave a characterization of the competition graphs of digraphs satisfying Condition $C(p)$ to show Theorem 1.1 as its corollary.

D. D. Scott [5] introduced the *competition-common enemy graph* of a digraph in 1987 as a variant of competition graph. The *competition-common enemy graph* of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices x and y if it holds that both $N_D^+(x) \cap N_D^+(y) \neq \emptyset$ and $N_D^-(x) \cap N_D^-(y) \neq \emptyset$, where $N_D^-(x) := \{v \in V(D) \mid (v, x) \in A(D)\}$ is the set of in-neighbors of x in D .

In this note, we characterize the competition-common enemy graphs of semiorders and interval orders as follows:

Theorem 1.2. *Let G be a graph. Then the following are equivalent.*

- (a) *G is the competition-common enemy graph of a semiorder,*

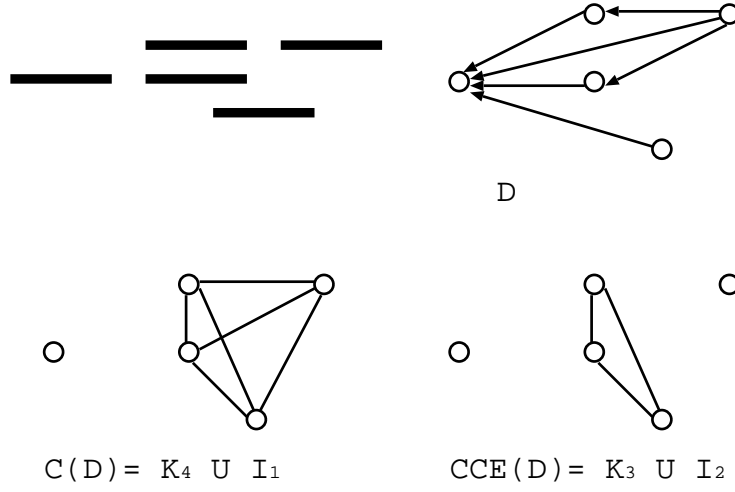


Figure 1: An interval order D , the competition graph $C(D)$, and the competition-common enemy graph $CCE(D)$

(b) G is the competition-common enemy graph of an interval order,

(c) $G = K_r \cup I_q$ where if $r \geq 2$ then $q \geq 2$. \square

Furthermore, we also characterize the competition-common enemy graphs of digraphs satisfying Conditions $C(p)$ and $C'(p)$.

2. Main Results

2.1. Conditions $C(p)$ and $C'(p)$

Definition. Let D be a digraph. For a set S of vertices in D , we define the following:

$$\begin{aligned}
 \mathcal{F}_D^+(S) &:= \{x \in S \mid N_D^+(x) \subseteq N_D^+(y) \text{ for all } y \in S\}, \\
 \mathcal{F}_D^-(S) &:= \{x \in S \mid N_D^-(x) \subseteq N_D^-(y) \text{ for all } y \in S\}, \\
 \mathcal{H}_D^+(S) &:= \{x \in S \mid N_D^+(x) \supseteq N_D^+(y) \text{ for all } y \in S\}, \\
 \mathcal{H}_D^-(S) &:= \{x \in S \mid N_D^-(x) \supseteq N_D^-(y) \text{ for all } y \in S\}.
 \end{aligned}$$

(Note that, in [3], an element in $\mathcal{F}_D^+(S)$ is called a *foot* of S and an element in $\mathcal{H}_D^+(S)$ is called a *head* of S .)

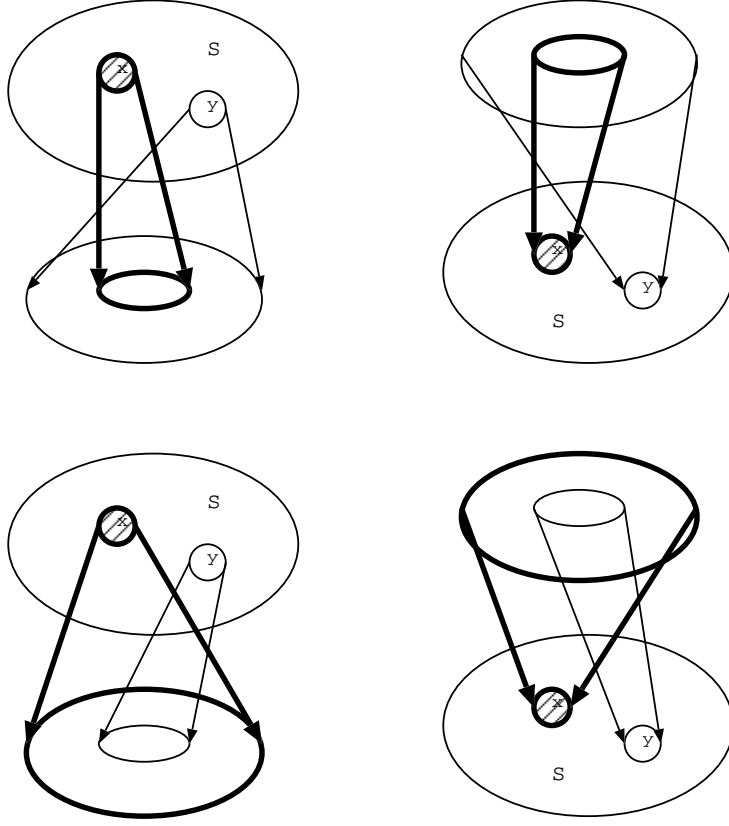


Figure 2: Elements x in $\mathcal{F}_D^+(S)$, $\mathcal{F}_D^-(S)$, $\mathcal{H}_D^+(S)$, and $\mathcal{H}_D^-(S)$

Let p be a positive integer with $p \geq 2$. We say that D satisfies *Condition $C(p)$* (resp. *Condition $C'(p)$* , *Condition $C^*(p)$* , *Condition $C^{*'}(p)$*) if the set $\mathcal{F}_D^+(S)$ (resp. $\mathcal{F}_D^-(S)$, $\mathcal{H}_D^+(S)$, $\mathcal{H}_D^-(S)$) is not empty for any set S of p vertices in the digraph D .

Proposition 2.1 ([3]). *Let $2 \leq p < q$. If a digraph D satisfies Condition $C(p)$, then the digraph D also satisfies Condition $C(q)$.*

Lemma 2.2. *Let D be a digraph and T, U be sets of vertices in D . If $\mathcal{F}_D^-(T) \cap U \neq \emptyset$, then $\mathcal{F}_D^-(U) \subseteq \mathcal{F}_D^-(T \cup U)$.*

Proof. Take $t \in \mathcal{F}_D^-(T) \cap U$. Then $N_D^-(t) \subseteq N_D^-(t')$ for any $t' \in T \setminus U$. If $\mathcal{F}_D^-(U)$ is empty, then the lemma trivially holds. So we assume that

$\mathcal{F}_D^-(U) \neq \emptyset$. Take any $u \in \mathcal{F}_D^-(U)$. Then $N_D^-(u) \subseteq N_D^-(u')$ for any $u' \in U$. Since $t \in U$, we have $N_D^-(u) \subseteq N_D^-(t)$. Therefore, $N_D^-(u) \subseteq N_D^-(t')$ for any $t' \in T \setminus U$. Thus $N_D^-(u) \subseteq N_D^-(s)$ for any $s \in (T \setminus U) \cup U = T \cup U$. Hence the lemma holds. \square

Proposition 2.3. *Let $2 \leq p < q$. If a digraph D satisfies Condition $C'(p)$, then the digraph D also satisfies Condition $C'(q)$.*

Proof. It is enough to show that D satisfies Condition $C'(p+1)$. Let S be any set of $p+1$ vertices of D , and let T be a subset of S with $|T| = p$. Then $\mathcal{F}_D^-(T) \neq \emptyset$ since D satisfies Condition $C'(p)$. Take an element x in $\mathcal{F}_D^-(T)$. Let U be a subset of S such that $|U| = p$ and $x \in U$. Since $p \geq 2$, it holds that $T \cup U = S$. By Lemma 2.2, we have $\mathcal{F}_D^-(U) \subseteq \mathcal{F}_D^-(T \cup U) = \mathcal{F}_D^-(S)$. Since D satisfies Condition $C'(p)$, $\mathcal{F}_D^-(U) \neq \emptyset$. Thus $\mathcal{F}_D^-(S)$ is not empty. \square

For a graph G , we denote the set of all isolated vertices in G by I_G . Then the graph $G - I_G$ is the union of the nontrivial connected components of G .

Proposition 2.4. *Let G be the competition-common enemy graph of a digraph D which satisfies Conditions $C(p)$ and $C'(p)$ for some $p \geq 2$. Suppose that $G - I_G$ has at least p vertices. Then $G - I_G$ is a clique of G .*

Proof. Take any two vertices a and b in $G - I_G$. Then a and b are not isolated. Let S be a set of p vertices in $G - I_G$ containing the vertices a and b . Since D satisfies Conditions $C(p)$ and $C'(p)$, there exist $x \in \mathcal{F}_D^+(S)$ and $y \in \mathcal{F}_D^-(S)$. Note that x and y are not isolated vertices. Take $u \in N_D^+(x)$ and $v \in N_D^-(y)$. By Condition $C(p)$, we have $u \in N_D^+(a) \cap N_D^+(b)$. By Condition $C'(p)$, we have $v \in N_D^-(a) \cap N_D^-(b)$. Therefore a and b are adjacent in $G - I_G$. Hence the proposition holds. \square

2.2. Classification

Theorem 2.5. *Let G be a graph and $p \geq 2$. Suppose that $G - I_G$ has at least p vertices. Then G is the competition-common enemy graph of a loopless digraph satisfying Conditions $C(p)$ and $C'(p)$ if and only if $G = K_r \cup I_q$ with $r \geq p$ and $q \geq 2$.*

Proof. First, we show the “only if” part. Let G be the competition-common enemy graph of a loopless digraph D satisfying Conditions $C(p)$ and $C'(p)$. Proposition 2.4 shows that $G = K_r \cup I_q$ with $r \geq p$ and $q \geq 0$. Suppose that $q = 0$ or $q = 1$. Since $r \geq p$, by Propositions 2.1 and 2.3, D satisfies Conditions $C(r)$ and $C'(r)$. Let $x \in \mathcal{F}_D^+(S)$ and $y \in \mathcal{F}_D^-(S)$ where $S :=$

$V(G - I_G)$. Since x and y are not isolated in G , we have $N_D^+(x) \neq \emptyset$ and $N_D^-(y) \neq \emptyset$. Let $u \in N_D^+(x)$ and $v \in N_D^-(y)$. If $u = v$, then $(s, u) \in A(D)$ and $(u, s) \in A(D)$ for any $s \in S$. Let S' be a set of p vertices containing the vertex u . Note that $S' \setminus \{u\} \subseteq S$ since $q \leq 1$. By Condition $C(p)$, $\mathcal{F}_D^+(S') \neq \emptyset$. If $u \in \mathcal{F}_D^+(S')$, then we have $s \in N_D^+(u) \subseteq N_D^+(s)$ for $s \in S' \setminus \{u\}$, i.e., $(s, s) \in A(D)$, which contradicts that D is loopless. If $s \in \mathcal{F}_D^+(S')$ for some $s \in S' \setminus \{u\}$, then we have $u \in N_D^+(s) \subseteq N_D^+(u)$, i.e., $(u, u) \in A(D)$, which contradicts that D is loopless. Therefore u and v must be distinct. Since $q \leq 1$, at least one of u and v is in $S = V(G - I_G)$. If $u \in S$, then we have $u \in N_D^+(x) \subseteq N_D^+(u)$, i.e., $(u, u) \in A(D)$. If $v \in S$, then we have $v \in N_D^-(y) \subseteq N_D^-(v)$, i.e., $(v, v) \in A(D)$. In any case, we reach a contradiction. Thus we have $q \geq 2$.

Second, we show the “if” part. Let $G = K_r \cup I_q$ with $r \geq p$ and $q \geq 2$. Let a and b be distinct vertices in I_q . We define a digraph D by $V(D) := V(G)$ and $A(D) := \{(a, x) \mid x \in V(K_r)\} \cup \{(x, b) \mid x \in V(K_r)\} \cup \{(a, b)\}$. Then D is loopless, D satisfies Conditions $C(p)$ and $C'(p)$, and the competition-common enemy graph of D is equal to G . \square

The *double competition number* $dk(G)$ of a graph G is the minimum number k such that G with k new isolated vertices is the competition-common enemy graph of an acyclic digraph.

Theorem 2.6. *Let G be a graph and $p \geq 2$. If G is the competition-common enemy graph of an acyclic digraph D satisfying Conditions $C(p)$ and $C'(p)$, then G is one of the following graphs:*

- (a) I_q ($q \geq 1$),
- (b) $K_r \cup I_q$ ($r \geq p$, $q \geq 2$),
- (c) $H \cup I_q$ where $|V(H)| < p$, $I_H = \emptyset$, and $q \geq dk(H)$.

Proof. Let G be the competition-common enemy graph of an acyclic digraph D satisfying Conditions $C(p)$ and $C'(p)$. If there is no nontrivial connected component in G , then (a) holds. Let H be the union of all nontrivial connected components of G . Then we have $G = H \cup I_q$ with $q \geq 0$ and $I_H = \emptyset$. If H has at least p vertices, then it follows from Theorem 2.5 that $G = K_r \cup I_q$ with $r \geq p$ and $q \geq 2$, i.e., (b) holds. Suppose that the number of the vertices of H is less than p . Since G is the competition-common enemy graph of an acyclic digraph D , the double competition number $dk(G)$ of G is equal to 0. Therefore, there must be at least $dk(H)$ vertices in I_q . Hence (c) holds. \square

2.3. Proof of Theorem 1.2

Proof of Theorem 1.2. (a) \Rightarrow (b): Since semiorders are a special case of interval orders where every interval has the same length, (a) implies (b).

(b) \Rightarrow (c): We can easily check that any interval order satisfies Conditions $C(2)$ and $C'(2)$. By Theorem 2.6 with $p = 2$, we can conclude that if (b) then (c).

(c) \Rightarrow (a): Suppose that $G = I_q$ ($q \geq 1$) or $G = K_r \cup I_q$ ($r \geq 2, q \geq 2$). When $G = I_q$, we let $f_1(x) := 0$ for any $x \in V(G)$ and let $\delta_1 := 1$. Then G is the competition-common enemy graph of the semiorder defined by f_1 and δ_1 . When $G = K_r \cup I_q$, we take a vertex a in I_q , let $f_2(x) := 0$ for any $x \in V(K_r)$, $f_2(a) := 2$, and $f_2(b) := -2$ for any $b \in V(I_q) \setminus \{a\}$, and let $\delta_2 := 1$. Then G is the competition-common enemy graph of the semiorder defined by f_2 and δ_2 .

Hence Theorem 1.2 holds. \square

3. Concluding Remarks

In this section, we present some problems for further study.

In Theorem 2.5, we gave a characterization of the competition common-enemy graphs G of digraphs satisfying Conditions $C(p)$ and $C'(p)$ if the number of the vertices of $G - I_G$ is at least p .

Problem 3.1. *Characterize the competition-common enemy graphs G of digraphs satisfying Conditions $C(p)$ and $C'(p)$ when the number of the vertices of $G - I_G$ is less than p .*

In this note, we didn't consider Conditions $C^*(p)$ and $C^{*'}(p)$.

Problem 3.2. *Characterize the competition-common enemy graphs of digraphs satisfying Conditions $C^*(p)$ and $C^{*'}(p)$.*

Niche graphs are another variant of competition graphs and were introduced by C. Cable, K. F. Jones, J.R. Lundgren, and S. Seager [1]. The *niche graph* of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices x and y if $N_D^+(x) \cap N_D^+(y) \neq \emptyset$ or $N_D^-(x) \cap N_D^-(y) \neq \emptyset$.

Problem 3.3. *What are the niche graphs of digraphs satisfying Conditions $C(p)$, $C'(p)$, $C^*(p)$, or $C^{*'}(p)$?*

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